SCOTCH COLLEGE

Practise Examination One

MATHEMATICS: SPECIALIST

Question/Answer Booklet – Section 1 – Calculators *NOT* allowed – Notes sheets *NOT* allowed

Teacher's Name: _____

Time allowed for this paper

Section	Reading	Working
Calculator-free	5 minutes	50 minutes
Calculator-assumed	10 minutes	100 minutes

Materials required/recommended for this paper

Section One (Calculator-free): 50 marks

To be provided by the supervisor
Section One Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: nil

Section Two (Calculator-assumed): 100 marks

To be provided by the supervisor
Section Two Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the School Curriculum and Standards Authority for this course.

Important Note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

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Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

Structure of this paper

Questions	Marks available	Your score
1	11	
2	8	
3	10	
4	6	
5	8	
6	7	
Total:	**50**	
8	6	
9	11	
10	8	
11	7	
12	10	
13	8	
14	8	
15	11	
16	9	
17	4	
18	6	
19	7	
20	5	
Total:	**100**	
Total marks = 150		
		%

See next page

Section One: Calculator-free **(50 Marks)**

This section has **six (6)** questions. Answer **all** questions. Write your answers in the space provided.

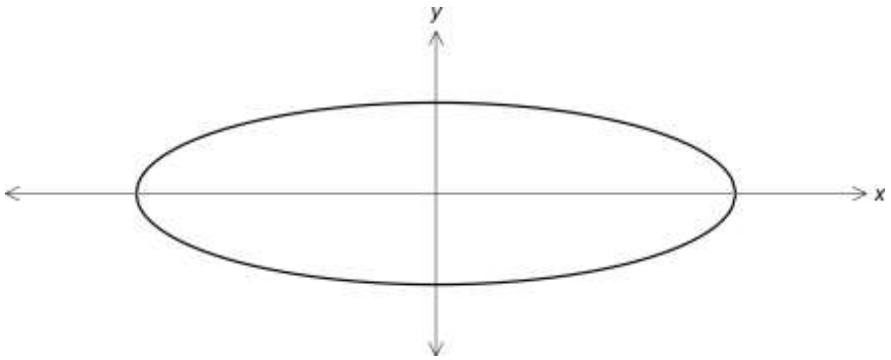
Working time: 50 minutes.

Question 1 **(11 marks)**

Consider the ellipse given by the following equation $\frac{x^2}{9}+\frac{y^2}{4}=1$

(a) Find the volume of the solid of revolution that is generated by rotating about the *x*-axis and express your answer in terms of π.

(4)



$$\frac{x^2}{9}+\frac{y^2}{4}=1$$
$$4x^2+9y^2=36$$
$$y^2=\frac{1}{9}(36-4x^2)$$

$$V=\pi\int y^2dx$$
$$V=\frac{1}{9}\pi\int_{-3}^{3}(36-4x^2)dx$$
$$V=\frac{1}{9}\pi\left[36x-\frac{4}{3}x^3\right]_{-3}^{3}$$
$$V=\frac{1}{9}\pi\left[36(3)-\frac{4}{3}(3)^3-36(-3)+\frac{4}{3}(-3)^3\right]$$
$$V=\frac{1}{9}\pi\left[24(9)-2(4)(9)\right]$$
$$V=16\pi\ units^3$$

(b) Explain why the equation given by the ellipse is not a function. What restriction would you place on the equation to make it a function?

(2)

The function is a one to many and fails the vertical line test.

$y \geq 0$

The path of a particle creates the ellipse and takes 12 seconds to complete a loop, starting at the position vector 3i metres moving in an anticlockwise direction.

(c) Find the vector equation of the position vector at any time, t, seconds.

(3)

$$r_t = 3\cos\left(\frac{\pi t}{6}\right)i + 2\sin\left(\frac{\pi t}{6}\right)j$$

(d) Find the maximum velocity the particle reaches and the time it takes.

(2)

The maximum velocity is 3 m/s and occurs at t=9 seconds

Question 2 **(8 marks)**

Two complex numbers are given by $w = 2cis\frac{\pi}{3}$ and $z = \sqrt{3} - i$.

(a) Express in polar form

(i) z (1)

$2cis\left(-\frac{\pi}{6}\right)$ ✓

(ii) $w \cdot z$ (1)

$4cis\frac{\pi}{6}$ ✓

(b) State the

(i) argument of $\frac{z}{w}$ (1)

$-\frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{2}$ ✓

(ii) modulus of $\frac{1}{z}$ (1)

$\frac{1}{2}$ ✓

(c) Show that the product $w \cdot \overline{w}$ is purely real. (2)

$$2cis\left(\frac{\pi}{3}\right) \times 2cis\left(-\frac{\pi}{3}\right) = 4cis0$$
$$= 4\cos(0) + 4i\sin(0)$$
$$= 4(1) + 4i(0)$$
$$= 4$$

✓ recognises $\overline{w}$ has a negative
✓
✓ Evalutes down to a real number

(d) For what positive values of n is v_n purely real, given $v_{n+1} = v_n \times w$, $v_1 = w$? (2)

$v_2 = 4cis\left(\frac{2\pi}{3}\right)$

$v_3 = 8cis\left(\frac{3\pi}{3}\right) = -8$

$v_4 = \ldots$

When $n = 3, 6, 9, 12, \ldots$

Hence n must be a multiple of 3.

✓ looks for a pattern

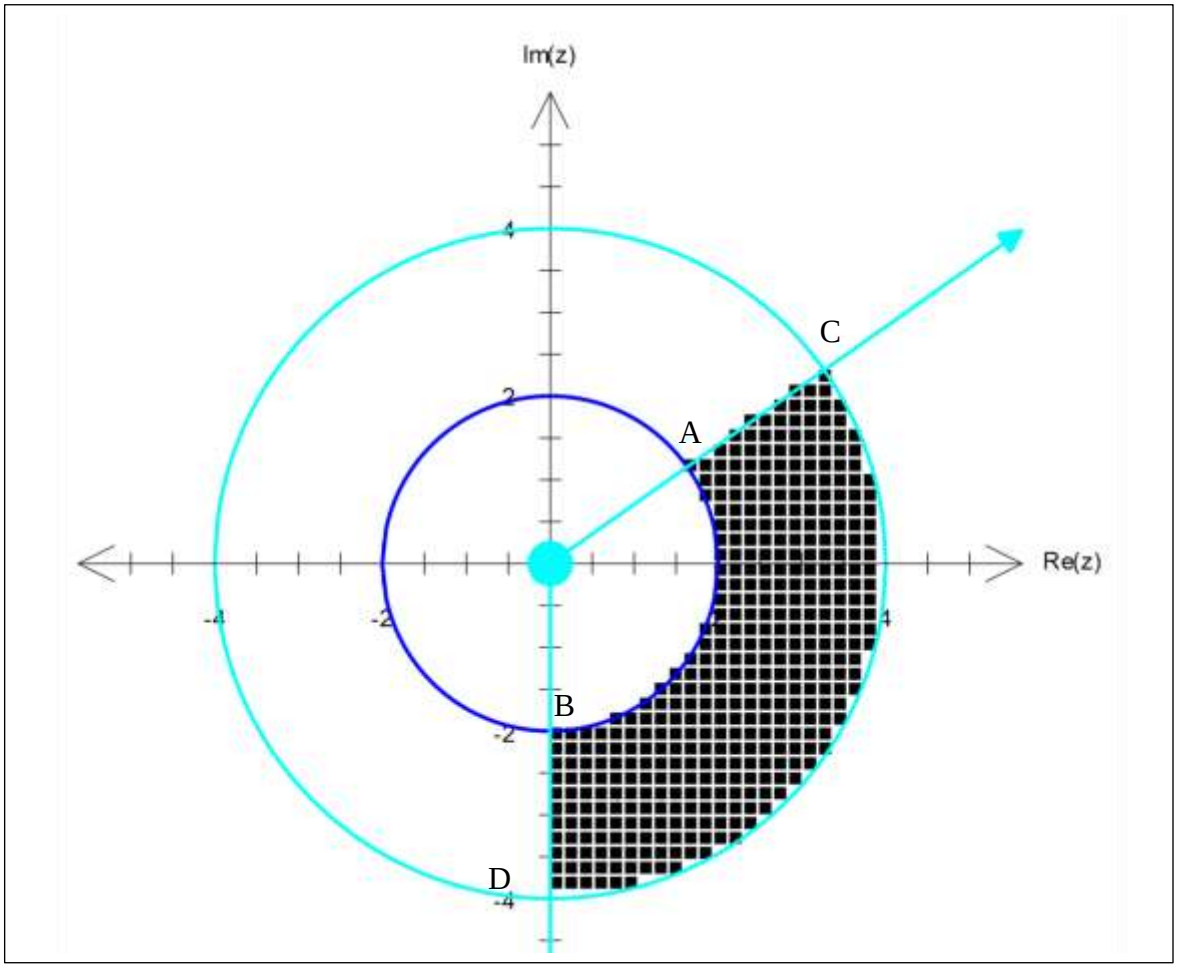
✓ States the pattern

See next page

Question 3 **(10 marks)**

(a) (i) Sketch the region of the complex plane defined by:

$$\left\{ z : \frac{-\pi}{2} \le \arg z \le \frac{\pi}{4} \quad \textit{and} \quad 2 \le |z| \le 4 \right\}$$ [3]



(ii) On your diagram above, label the points $A\left(2, \frac{\pi}{4}\right)$, $B\left(2, -\frac{\pi}{2}\right)$, $C\left(4, \frac{\pi}{4}\right)$, $D\left(4, -\frac{\pi}{2}\right)$. [1]

(iii) Find the exact distance from A to D. [3]

$$AD^2 = 2^2 + 4^2 - 2(2)(4)\cos\left(\frac{3\pi}{4}\right)$$

$$AD^2 = 20 + 16\cos\left(\frac{\pi}{4}\right)$$

$$AD = \sqrt{20 + 8\sqrt{2}}$$

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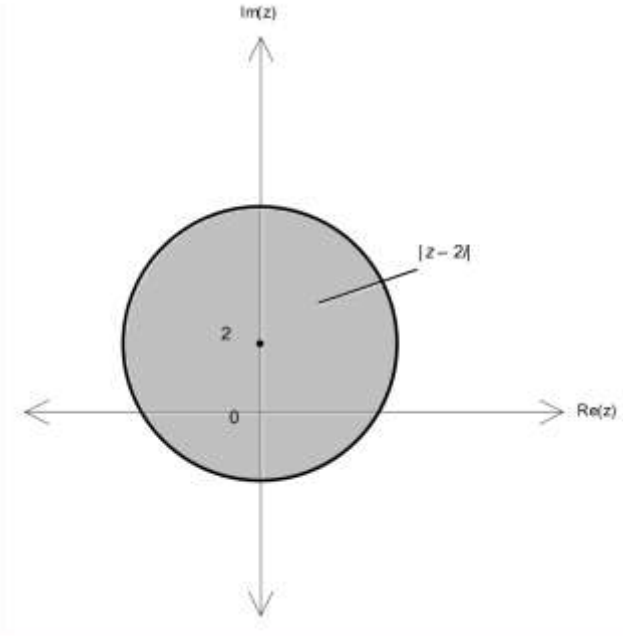
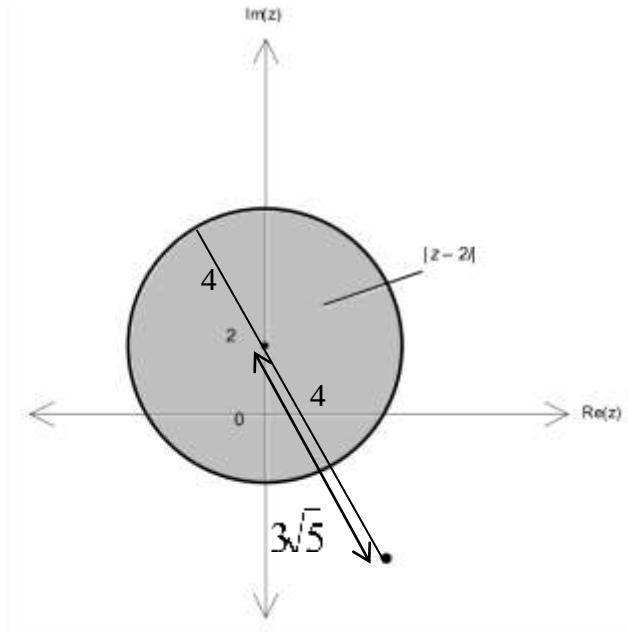
Question 3 *continued*

A locus of points defined by $|z-2i| \leq 4$ is represented on the Argand diagram below.



(b) find the exact greatest and least values of $|z-3+4i|$ in relation to the locus of points defined by $|z-2i| \leq 4$. [3]



$3-4i-2i$

$=3-6i$

$=\sqrt{3^2+6^2}$

$=\sqrt{45}$

$=3\sqrt{5}$

Minimum Distance

$3\sqrt{5}-4$

Maximum Distance

$3\sqrt{5}+4$

See next page

Question 4 **(6 marks)**

Evaluate $\int \frac{5x^2+1}{x^3+x}dx$

$$\int \frac{5x^2+1}{x^3+x}dx$$

$$\frac{A}{x}+\frac{Bx}{x^2+1}$$

$$A(x^2+1)+Bx(x)=5x^2+1$$

$$Let\ x=0$$

$$A=1$$

$$(x^2+1)+Bx^2=5x^2+1$$

$$(1+B)=5$$

$$B=4$$

$$\int \frac{1}{x}dx+\int \frac{4x}{x^2+1}dx$$

$$=\ln|x|+2\ln|x^2+1|+c$$

$$=\ln kx(x^2+1)^2$$

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Question 5 **(8 marks)**

(a) Integrate $\tan x\left(1+\cos^2 x\right)dx$ [4]

$$\begin{aligned}
&\int \tan x\left(1+\cos^2 x\right)dx \\
&= \int \frac{\sin x}{\cos x}\left(1+\cos^2 x\right)dx \\
&= \int\left(\frac{\sin x}{\cos x}+\cos x\right)dx \\
&= \int \frac{\sin x}{\cos x}dx + \int \cos x\,dx \\
&= -\ln\cos x + \sin x + c \\
&= \sin x - \ln\cos x + c
\end{aligned}$$

(b) Solve $\frac{dy}{dx}=\frac{x+1}{y^4+1}$, given that when $x=2,\ y=1$. [4]

$$\begin{aligned}
&\frac{dy}{dx}=\frac{x+1}{y^4+1} \\
&\int (y^4+1)dy = \int (x+1)dx \\
&\frac{y^5}{5}+y=\frac{x^2}{2}+x+c \\
&x=2 \quad y=1 \\
&\frac{1^5}{5}+1=\frac{2^2}{2}+2+c \\
&c=\frac{-14}{5} \\
&\frac{y^5}{5}+y=\frac{x^2}{2}+x+\frac{-14}{5} \\
&2y^5+10y=5x^2+10x-28
\end{aligned}$$

Question 6 **(7 marks)**

Two particles, A and B, are tracked in a laboratory experiment.

Particle A was observed to have initial position $2\mathbf{i} - \mathbf{j} + 8\mathbf{k}$ cm and constant velocity vector $-2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ cm/s.

Particle B was observed to have initial position $-2\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}$ cm and constant velocity vector $3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ cm/s.

If the particles continue with these velocities, find the minimum distance between them in the subsequent motion and the time when this occurs.

Let $\mathbf{v}$ and $\mathbf{r}$ be velocity and displacement of A relative to B

$$\mathbf{v} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ -6 \end{bmatrix}$$ ✓

$$\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 17 \end{bmatrix}$$ ✓

$$t\mathbf{v} + \mathbf{r} = \begin{bmatrix} -5t + 4 \\ 2t - 4 \\ -6t + 17 \end{bmatrix}$$ ✓

$$\begin{bmatrix} -5 \\ 2 \\ -6 \end{bmatrix} \bullet \begin{bmatrix} -5t + 4 \\ 2t - 4 \\ -6t + 17 \end{bmatrix} = 25t - 20 + 4t - 8 + 36t - 102$$ ✓

$65t - 130 = 0$ when $t = 2$ seconds ✓

$$2\mathbf{v} + \mathbf{r} = \begin{bmatrix} -10 + 4 \\ 4 - 4 \\ -12 + 17 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 5 \end{bmatrix}$$

✓ for recognising it is magnitude
✓ for stating the correct length with units

$| 2\mathbf{v} + \mathbf{r} | = \sqrt{36 + 25} = \sqrt{61}$ cm

End of questions

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